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ON A METHOD OF EXTREMAL CONTROL*

M.I. LOGINOV

An encounter game problem /1,2/ is analyzed on a prescribed time interval for controlled objects whose dynamics are described by nonlinear differential equations. It is assumed that the game's payoff is a convex function, differentiable in some domain, of the difference between the objects' final states. Under specific conditions a procedure is justified for the formation of an extremal strategy of one of the players, guaranteeing him a game result no worse than in the corresponding programmed maximin problem for the initial position. By example it is shown that in the case of nonlinear systems the procedure described in the paper for constructing the optimal strategy covers certain irregular situations in which the extremal aiming rule developed for linear /1/ and nonlinear /3/ controlled systems is inapplicable. In the case of linear systems the conditions found in the paper ensure the regularity of the encounter game problem and, as shown in /4/, the method proposed for solving the encounter problem occupies an intermediate position between the extremal aiming rule /1,2/ and the direct methods in differential game theory /2,5/.

1. Consider the motions $y(t)$ and $z(t)$ of controlled objects described by the nonlinear differential equations

$$y^{\cdot} = f^{(1)}(t, y) + g^{(1)}(t, u),\ u \in P,\ y \in R^n \qquad (1.1)$$
$$z^{\cdot} = f^{(2)}(t, z) + g^{(2)}(t, v),\ v \in Q,\ z \in R^n$$

(the sets P and Q are compacta in R^p and R^q, respectively). We assume that the motions $y(t)$ and $z(t)$ are examined on a prescribed interval $[t_0, \vartheta]$ and that the payoff is determined by the equality

$$\gamma[\vartheta] = \sigma(\{z(\vartheta)\}_m - \{y(\vartheta)\}_m) = \sigma(x(\vartheta))$$

where $\sigma(x)$ is a prescribed function of the vector-valued argument x; $\{z\}_m$, $\{y\}_m$ are vectors composed of the first m components of vectors z and y. Having the choice of the control $u \in P$ ($v \in Q$) at his disposal, the first (second) player tries to minimize (maximize) the quantity $\gamma[\vartheta]$. By $U(\cdot \mid t, \vartheta)$ and $V(\cdot \mid t, \vartheta)$ we denote the sets of Borel-measurable functions $u(\cdot): T \to P$ and $v(\cdot): T \to Q$, where $T = [t, \vartheta]$; by $y(\tau; t, y, u(\cdot))$ and $z(\tau; t, z, v(\cdot))$, $\tau \in T$ we denote the solutions of Eqs.(1.1) generated by the controls $u(\cdot)$ and $v(\cdot)$ under the initial conditions $y(t) = y$, $z(t) = z$. Let

$$\rho_1(t, \vartheta, y, l) = \sup_{U(\cdot \mid t, \vartheta)} l' \{y(\vartheta; t, y, u(\cdot))\}_m \qquad (1.2)$$
$$\rho_2(t, \vartheta, z, l) = \sup_{V(\cdot \mid t, \vartheta)} l' \{z(\vartheta; t, z, v(\cdot))\}_m$$

where l is an arbitrary nonzero m-dimensional vector; the prime denotes transposition.

Condition 1. A. The functions $f^{(i)}, g^{(i)}$ $(i = 1, 2)$ are continuous in all variables, while the functions $f^{(i)}$ $(i = 1, 2)$ are continuously differentiable in the variables y and z, respectively, and satisfy the conditions

$$x' f^{(i)}(t, x) \leqslant c(\|x\|^2 + 1) \quad (i = 1, 2;\ c = \text{const})$$

B. The function $\sigma(x)$ is convex and has continuous and uniformly bounded derivatives in domain $G = \{x \mid \sigma(x) > \inf_z \sigma(z)\}$.

C. The sets

$$Q_1(t) = \{g^{(1)} \mid g^{(1)} = g^{(1)}(t, u),\ u \in P\},\quad Q_2(t) = \{g^{(2)} \mid g^{(2)} = g^{(2)}(t, v),\ v \in Q\}$$

are convex for all $t \in [t_0, \vartheta]$

D. For any unit vector l the maximum in the right-hand sides of equalities (1.2) is reached on a unique programmed motion $\{y^\circ(\tau; t, y, l), z^\circ(\tau; t, z, l)\}$ generated by the vector-valued functions $\{g^{(1)}(\tau; u^\circ(\tau; t, y, l)), g^{(2)}(\tau; v^0(\tau; t, z, l))\}$.

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We note that when Condition 1 is fulfilled the attainability domains $G_1(t, \vartheta, y)$ and $G_2(t, \vartheta, z)$ of motions $\{y(\tau; t, y, u(\cdot))\}_m$ and $\{z(\tau; t, z, v(\cdot))\}_m$, corresponding to the initial position $\{t, y, z\}$, are convex campacta in R^m by the instant $\tau = \vartheta$ /3/, while the quantities ρ_1 and ρ_2 are the support functions of sets G_1 and G_2. Let (l) be the function adjoint /6,7/ to the convex function $\sigma(x)$, i.e., $\omega(l) = \sup_x \{l'x - \sigma(x)\}$ and $L = \mathrm{dom}\, \omega(\cdot) = \{l \in R^m \mid \omega(l) < \infty\}$. The equality

$$\sigma(x) = \max_{l \in L} \{l'x - \omega(l)\}$$

is valid /6/ on the strength of Condition 1B. We introduce into consideration the programmed maximin quantity for the initial position $\{t_0, y_0, z_0\}$

$$\varepsilon^\circ(t_0, y_0, z_0) = \max_{z(\vartheta)} \min_{y(\vartheta)} \sigma(\{z(\vartheta)\}_m - \{y(\vartheta)\}_m) =$$
$$\max_{z(\vartheta)} \min_{y(\vartheta)} \max_{l} \{l'\{z(\vartheta)\}_m - l'\{y(\vartheta)\}_m - \omega(l)\}$$
$$\{y(\vartheta)\}_m \in G_1(\vartheta; t_0, y_0),\ \{z(\vartheta)\}_m \in G_2(\vartheta; t_0, z_0),\ l \in L$$

The adjoint function $\omega(l)$ is convex; therefore, on the basis of the general minimax theorem /8/ we can write

$$\varepsilon^\circ(t_0, y_0, z_0) = \max_{l \in L} \{\rho_2(l, \vartheta, t_0, z_0) - \rho_1(l, \vartheta, t_0, y_0) - \omega(l)\} \tag{1.3}$$

Condition 2. Let $L^\circ = L^\circ(t_0, y_0, z_0)$ be the set of vectors l on which the maximum in the right-hand side of equality (1.3) is reached. At least one nonzero vector $l^\circ = l^\circ(t_0, y_0, z_0) \in L^\circ$ exists such that:

A. The derivatives

$$\partial y^\circ(\vartheta, t, y, l^\circ) / \partial y = Y[\vartheta; t, y, l^\circ],\ \partial z^\circ(\vartheta, t, z, l^\circ) / \partial z = Z[\vartheta; t, z, l^\circ]$$

continuous in all variables, exist for any position $\{t, y, z\}$.

B. The function

$$\varkappa(l, t, y, z) = \max_{u \in P} l'\{Y[\vartheta; t, y, l^0]\}'q^{(1)}(t, u) - \max_{v \in Q} l'\{Z[\vartheta; t, z, l^\circ]\}_m' q^{(2)}(t, v) \tag{1.4}$$

is convex in l for all $\{t, y, z\}$ where $\{Y\}_m$ and $\{Z\}_m$ are $n \times m$-matrices composed of the first m columns of matrices Y and Z.

C. For any absolutely continuous functions $y(t)$ and $z(t)$ and for almost all $t\ (t \in [t_0, \vartheta])$ the maxima in the equalities

$$l^{\circ\prime}\{Y[\vartheta; t, y(t), l^\circ]\}_m q_*^{(1)} = \max_{q \in Q_1(t)} l^{\circ\prime}\{Y[\vartheta; t, y(t), l^\circ]\}_m' q$$
$$l^{\circ\prime}\{Z[\vartheta; t, z(t), l^\circ]\}_m' q_*^{(2)} = \max_{q \in Q_2(t)} l^{\circ\prime}\{Z[\vartheta; t, z(t), l^\circ]\}_m' q$$

are reached on the unique vectors $q_*^{(1)} = q_*^{(1)}(t, y(t), l^\circ)$ and $q_*^{(2)} = q_*^{(2)}(t, z(t), l^\circ)$.

In the general case it is difficult to verify Conditions 1 and 2 for nonlinear systems. However, we can find requirements on the first-approximation system which ensure the fulfilment of Conditions 1 and 2 for the quasilinear controlled objects

$$y^\cdot = A^{(1)}(t)y + B^{(1)}(t)u + \lambda f^{(1)}(y, t),\ \|y\| \leqslant \mu \tag{1.5}$$
$$z^\cdot = A^{(2)}(t)z + B^{(2)}(t)v + \lambda f^{(2)}(z, t),\ \|v\| \leqslant \nu$$

where λ is a small parameter and the $f^{(i)}\ (i = 1, 2)$ are continuous in $t \in [t_0, \vartheta]$ and twice continuously differentiable in the phase variables. Let $Y[\vartheta, t]$ and $Z[\vartheta, t]$ be the fundamental matrices of the linear homogeneous systems corresponding to Eqs. (1.5) with $\lambda = 0,\ u = v = 0$. Then Conditions 1 and 2 can be replaced /9/ by the following two requirements:

1) for any unit vector l the functions

$$\xi^{(1)}(t) = \|l'\{Y[\vartheta, t] B^{(1)}(t)\}_m\|,\ \xi^{(2)}(t) = \|l'\{Z[\vartheta, t] B^{(2)}(t)\}_m\|$$

vanish at a finite number points $t_j^{(i)}$ (the $t_j^{(i)}$ are from the interval $[t_0, \vartheta]$), and

$$|d\xi^{(i)}/dt|_{t=t_j^{(i)}} \geqslant k > 0 \quad (k = \mathrm{const},\ i = 1, 2)$$

2) the function

$$\varkappa(l, t) = \max_{\|u\| \leqslant \mu} l' \{Y[\vartheta, t] B^{(1)}(t)\}_m u - \max_{\|v\| \leqslant \nu} l' \{Z[\vartheta, t] B^{(2)}(t)\} v$$

satisfies the inequality

$$\varkappa(l_1, t) + \varkappa(l_2, t) > \varkappa(l_1 + l_2, t), \ \forall t \in [t_0, \vartheta]$$

for any two vectors l_1 and l_2 $(l_1 \neq R l_2,\ R = \text{const})$.

The first player's admissible strategy U is defined as a many valued mapping associating with each possible position $\{t, y, z\}$ a set $U(t, y, z) \subset P$ upper-semicontinuous with respect to inclusion. The function $g^{(1)}$ is continuous; therefore, the closed convex hull of the set

$$Q_1(t, y, z, U) = \{g^{(1)} \mid g^{(1)} = g^{(1)}(t, u), u \in U(t, y, z)\}$$

is upper-semicontinuous with resepct to inclusion as well. By motions $y[t]$ we mean the solutions of the corresponding contingency equations /1,2,10/. Let $Y(t_0, y_0, z_0\ U)$ and $Z(t_0, z_0)$ be sets of solutions corresponding to the initial position $\{t_0, y_0, z_0\}$ of the contingency equations

$$y^{\cdot} \in f^{(1)}(t, y) + \operatorname{co} Q_1(t, y, z, U),\ z^{\cdot} \in f^{(2)}(t, z) + Q_2(t)$$

Problem. Find the first player's admissible strategy U° ensuring the equality

$$\max_{y[\cdot]} \max_{z[\cdot]} \{\sigma(\{z[\vartheta]\}_m - \{y[\vartheta]\}_m) \mid y[\cdot] \in Y(t_0, y_0, z_0, U^\circ),$$
$$z[\cdot] \in Z(t_0, z_0)\} = \min_U \operatorname{Idem}(U^\circ \to U)$$

Here and later the Idem in an equality's right-hand side denotes an expression coinciding with the left-hand side of this equality under the change of symbols indicated within the parentheses.

2. We introduce into consideration the function

$$\varepsilon(t, y, z) = \sigma(x(\vartheta, t, y, z, l^\circ)) = \sigma(\{z^\circ(\vartheta, t, z, l^\circ)\}_m - \{y^0(\vartheta, t, y, l^\circ)\}_m)$$

where $l^\circ = l^\circ(t_0, y_0, z_0)$ is the vector figuring in Condition 2.

Definition. Let an m-dimensional vector $s(t, y, z)$ be defined by the equality

$$s(t, y, z) = -\partial\sigma(x(\vartheta; t, y, z, l^\circ))/\partial x$$

The set $U^*(t, y, z)$ specifying the strategy U^* consists of all vectors $u^* \in P$ for which the condition

$$s'(t, y, z)\{Y[\vartheta; t, y, l^\circ]\}_m' g^{(1)}(t, u^*) = \min_{u \in P} \operatorname{Idem}(u^* \to u)$$

is fulfilled.

Theorem. If Conditions 1 and 2 are fulfilled, then the strategy constructed in the Definition guarantees the first player the result

$$(\gamma[\vartheta] \mid t_0, y_0, z_0, U^*) \leqslant \varepsilon^\circ(t_0, y_0, z_0)$$

and, consequently, is the optimal strategy solving the problem.

Proof. Suppose that the position $\{t, y[t], z[t]\}$ is realized and let $x(\vartheta; t, y[t], z[t], l^\circ) \in G$. From Condition 2 follows the existence of the derivatives

$$\frac{\partial \varepsilon}{\partial y} = -\{Y[\vartheta; t, y, l^\circ]\}_m \frac{\partial \sigma}{\partial x},\ \frac{\partial \varepsilon}{\partial z} = \{Z[\vartheta; t, z, l^\circ]\} \frac{\partial \sigma}{\partial x}$$

Let us estimate the increment of the function $\varepsilon[\tau] = \varepsilon(\tau, y[\tau], z[\tau])$ on the interval $[t, t + \Delta t]$. We write it as $\Delta\varepsilon = \delta_1 + \delta_2$

$$\delta_1 = \varepsilon(t + \Delta t, y[t + \Delta t], z[t + \Delta t]) - \varepsilon(t + \Delta t, y[t], z[t])$$
$$\delta_2 = \varepsilon(t + \Delta t, y[t], z[t]) - \varepsilon(t, y[t], z[t])$$

With due regard to the continuity of the derivatives $\partial\varepsilon/\partial y$ and $\partial\varepsilon/\partial z$ with respect to $\{t, y, z\}$ we have

$$\delta_1 = s'(t, y[t], z[t])\{Y[\vartheta; t, y[t], l^\circ]\}_m' \times \tag{2.1}$$
$$\int_t^{t+\Delta t} [f^{(1)}(\tau, y[\tau]) + q_1[\tau]]\, d\tau - s'(t, y[t], z[t])\{Z[\vartheta; t, z[t], l^\circ]\}_m' \times$$

$$\int_t^{t+\Delta t} [f^{(2)}(\tau, z[\tau]) + q^{(2)}(\tau, v[\tau])]\, dr + o(\Delta t);$$

$$q_1[\tau] \in \text{co}\, Q_1(\tau, y[\tau], z[\tau], U)$$

Allowing for the uniqueness of the programmed motions $y^\circ(\cdot\ t,\ y,\ l^\circ)$ and $z^\circ(\cdot\ t,\ z,\ l^\circ)$, we obtain

$$\begin{gathered}\delta_2 = \varepsilon(t+\Delta t, y[t], z[t]) - \varepsilon(t+\Delta t, y^0(t+\Delta t; t, y[t], l^\circ), \\ z^\circ(t+\Delta t; t, z[t], l^\circ) = -s'(t, y[t], z[t])\{Y[\vartheta; t, \\ y[t], l^\circ]\}_m' \int_t^{t+\Delta t} [f^{(1)}(\tau, y^\circ(\tau; t, y[t], l^\circ)) + q^{(1)}(\tau; u^\circ(\tau; t, y[t], l^\circ))]\, d\tau + \\ s'(t, y[t], z[t])\{Z[\vartheta; t, z[t], l^\circ]\}_m \times \\ \int_t^{t+\Delta t} [f^{(2)}(\tau, z^\circ(\tau; t, z[t], l^\circ)) + q^{(2)}(\tau, v^\circ(\tau; t, z[t], l^\circ))]\, d\tau + o(\Delta t)\end{gathered} \tag{2.2}$$

The function $\varkappa(l, t, y, z)$ of (1.4) is convex in l and positive-homogeneous; therefore it is the support function of the nonempty convex set

$$H(t, y, z) = \bigcap_{v \in Q} [\{Y[\vartheta; t, y, l^\circ]\}_m' Q_1(t) - \{Z[\vartheta; t, z, l^\circ]\}_m' q^{(2)}(t, v)]$$

Relying on Pontriagin's maximum principle and on the results in /3/ (or on the dynamic programming method), with due regard to Condition 2C it can be shown that for almost all $t \in [t_0, \vartheta]$ the inclusion

$$\begin{gathered}h^\circ(t, y[t], z[t]) = \{Y[\vartheta; t, y[t], l^\circ]\}_m' q_*^{(1)} - \\ \{Z[\vartheta; t, z[t], l^\circ]\}_m' q_*^{(2)} \in H(t, y[t], z[t])\end{gathered}$$

where $q_*^{(1)} = q_*^{(1)}(t, y, [t], l^\circ)$, $q_*^{(2)} = q_*^{(2)}(t, z[t], l^\circ)$, is valid. From (2.1) and (2.2), the continuity of vector s and of the matrices Y and Z, and the fact that the vector-valued functions $q^{(1)}(\tau, u^\circ(\tau; t, y[t], l^\circ))$ and $q^{(2)}(\tau; v^\circ(\tau; t, z[t], l^\circ))$ satisfy the maximum condition, we obtain

$$\begin{gathered}\Delta\varepsilon = \int_t^{t+\Delta t} s'(\tau, y[\tau], z[\tau])[\{Y[\vartheta; \tau, y[\tau], l^\circ]\}_m' q_1[\tau] - \\ \{Z[\vartheta; \tau, z[\tau], l^\circ]\}_m' q^{(2)}(\tau, v[\tau]) - h^\circ(\tau, y[\tau], z[\tau])]\, d\tau + o(\Delta t)\end{gathered}$$

Further, allowing for the form of the set $H(\tau, y[\tau], z[\tau])$, we conclude that from any realization $y[\tau]$ we can find $q = q(\tau)$ such that

$$\Delta\varepsilon = \int_t^{t+\Delta t} s'(\tau, y[\tau], z[\tau])\{Y[\vartheta; \tau, y[\tau], l^\circ]\}_m'(q_1[\tau] - q(\tau))d\tau + o(\Delta t)$$

$$q(\tau) \in Q_1(\tau)$$

Thus, by choosing $U = U^*$ we get that $\varepsilon[t]$ does not grow on the interval $[t_0, \vartheta]$. Now taking into account that the equalities

$$\begin{gathered}\varepsilon[t_0] = \varepsilon^\circ(t_0, y_0, z_0) \\ \varepsilon[\vartheta] = \sigma(\{z[\vartheta]\}_m - \{y[\vartheta]\}_m\end{gathered}$$

hold by the construction of the auxiliary function $\varepsilon[t]$, we conclude that the theorem's assertion is valid.

We note that if the function $\varkappa(l, t, y, z)$ is concave in l for all (t, y, z), then by interchanging the roles of u and v in the preceding arguments we can obtain a strategy V^* guaranteeing the second player the result

$$(\gamma[\vartheta] \mid t_0, y_0, z_0, V^*) \geqslant \varepsilon^\circ(t_0, y_0, z_0)$$

3. Example. Let the behaviors of the pursuer and evader object be described by the equations

$$\begin{gathered}y_1^{\cdot} = y_2,\ y_2^{\cdot} = \lambda y_2^3 + u_1,\ y_3^{\cdot} = y_4,\ y_4^{\cdot} = \lambda y_4^3 + u_2 \\ u_1^2 + u_2^3 \leqslant \mu^2,\ z_1^{\cdot} = z_2,\ z_2^{\cdot} = \lambda z_2^3 + v_1,\ z_3^{\cdot} = z_4 \\ z_4^{\cdot} = \lambda z_4^3 + v_2,\ v_1^2 + v_1^2 \leqslant \nu^2\end{gathered}$$

and let

$$\gamma[\vartheta] = [(y_1(\vartheta) - z_1(\vartheta))^2 + (y_2(\vartheta) - z_2(\vartheta))^2]^{1/2}, \ \lambda > 0$$

We denote by $\rho_1(l, t, y)$ and $\rho_2(l, t, z)$ the support functions of the attainability domains $G_1(\vartheta, t, y)$ and $G_2(\vartheta, t, z)$. Carrying out the necessary computations, analogous to those in /9/, we obtain

$$\begin{aligned}\varphi(l, t, y, z, \lambda) &= \rho_2(l, t, z) - \rho_1(l, t, y) = l_1(x_1 + (\vartheta - t)x_2 + \\ &\lambda(\vartheta - t)^2 (z_2^3 - y_2^3)/2) + l_2(x_3 + (\vartheta - t)x_4 + \lambda(\vartheta - t)(z_4^3 - \\ &y_4^3)/2) + l_1^2 [\lambda\alpha(\vartheta - t)^4/4 + \lambda(\vartheta - t)^3(z_2^2 - y_2^2)/2] + \\ &l_2^2 [\lambda\alpha(\vartheta - t)^4/4 + \lambda(\vartheta - t)^3 (z_4^2 - y_4^2)/2] + \\ &l_1^3\lambda(\vartheta - t)^4 (z_2 - y_2)/4 + l_2^3 \lambda(\vartheta - t)^4 (z_4 - y_4)/4 + \\ &\lambda^2 R(l_1, l_2, y, z, \lambda)\end{aligned}$$

$$(l_1^2 + l_2^2 = 1, x_i = z_i - y_i \ (i = 1, \ldots, 4), \ \mu = 1, \nu = 1 + \alpha(\vartheta - t)^2\lambda)$$

We select the initial position $\{t_0, y_0, z_0\}$ such that $t_0 = 0$, $y_{20} = y_{40}$, $z_{20} = z_{40}$, $y_{20} - z_{20} = a > 0$. We denote

$$\begin{aligned}c &= \vartheta^4 a/4, \ b = \vartheta^3 (z_{20}^2 - y_{20}^2)/2 + \alpha\vartheta^4/4 \\ a_1 &= x_{10} + \vartheta x_{20} + \lambda\vartheta^2 (z_{20}^3 - y_{20}^3)/2 \\ a_2 &= x_{30} + \vartheta x_{40} + \lambda\vartheta^2 (z_{40}^3 - y_{40}^3)/2\end{aligned}$$

and we select x_{10}, x_{30} such that

$$a_1 = {}^3/_4\,\lambda c - k_1\lambda^2, \ a_2 = {}^3/_4\lambda c - k_2\lambda^2$$

where the parameters k_1 and k_2 are defined below. Under the assumptions made

$$\rho_2 - \rho_1 = {}^3/_4\,\lambda c(l_1 + l_2) - \lambda c(l_1^3 + l_2^3) + \lambda b(l_1^2 + l_2^2) + \lambda^2 (R - k_1 l_1 - k_2 l_2)$$

We denote $l_1 = \cos\varphi$, $l_2 = \sin\varphi$, then

$$\varepsilon^\circ(t_0, y_0, z_0) = \max_{\|l\|=1} \{\rho_2 - \rho_1\} = \max_{\varphi} \Phi(\varphi, \lambda, k_1, k_2)$$

$$\Phi(\varphi, \lambda, k_1, k_2) = \frac{\lambda\sqrt{2}}{4} c \sin\left(3\varphi - \frac{\pi}{4}\right) + \lambda b + \lambda^2 (R - k_1\cos\varphi - k_2 \sin\varphi)$$

The sum of the first two summands is maximal for the following values of φ:

$$\varphi^{(0)} = \frac{\pi}{4}, \quad \varphi^{(1)} = \frac{11}{12}\pi, \quad \varphi^{(2)} = \frac{19}{12}\pi \tag{3.1}$$

and the magnitude of this maximum equals $\lambda\sqrt{2}c/4 + \lambda b$. Using the implicit function theorem we can show that for sufficiently small λ we can find parameters k_1 and k_2 such that the function Φ has precisely three local maxima for $\varphi^{(0)}(\lambda), \varphi^{(1)}(\lambda), \varphi^{(2)}(\lambda)$ corresponding to the values (3.1) when $\lambda = 0$, and they are equal. The quantity $\varepsilon^\circ(t_0, y_0, z_0)$ is positive if $\sqrt{2}\,c/4 + b > 0$ or, setting $\alpha = (1 + 2\sqrt{2})\,a$, we obtain the condition

$$a\vartheta^3 \left[\frac{\sqrt{2}}{16}\vartheta - \frac{y_{20} + z_{20}}{2} + \frac{1 + 2\sqrt{2}}{4}\vartheta\right] > 0 \tag{3.2}$$

The function $\varkappa(l, t, y, z, \lambda)$ computed for the vector $l^{(0)}(\lambda)$ $(l^{(0)}(0) = (\sqrt{2}/2, \sqrt{2}/2)$ has the form

$$\begin{aligned}\varkappa = \frac{\lambda(\vartheta - t)^2}{\sqrt{l_1^2 + l_1^2}} &\Big[({}^3/_2(y_2^2 - z_2^2) + \frac{\sqrt{2}}{2}(\vartheta - t)(y_2 - z_2) - \\ &(1 + 2\sqrt{2})\,a(\vartheta - t))\,l_1^2 + ({}^3/_2(y_4^2 - z_4^2) + \frac{\sqrt{2}}{2}(\vartheta - t)(y_4 - z_4) - \\ &(1 + 2\sqrt{2})\,a(\vartheta - t))\,l_2^2\Big] + o(\lambda)\end{aligned}$$

This function is convex for all realizations $y[t], z[t]$ $(0 \leqslant t < \vartheta)$ if a sufficiently large and

$$y_{20} + z_{20} > \left[{}^2/_3(1 + 2\sqrt{2}) - \frac{\sqrt{2}}{3}\right]\vartheta \tag{3.3}$$

Conditions (3.2) and (3.3) are fulfilled if $y_{20} + z_{20} = 2.09\,\vartheta$. Thus, we have found and initial position for which there is no regularity in the sense of /1,3/, but the method described in the paper remains applicable.

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